Score Driven Models and their Diffusion Limits

April 16, 2022

1 Introduction

Score driven models are a class of models that has been introduced into the econometric literature by the independent work of [Harvey, 2013] and [Creal et al., 2013]. The aim was to provide a unified and encompassing framework for a number of observation driven models that showed success in applied settings such as the GARCH model specified by [Bollerslev, 1987]. The key feature of score driven models is that the dynamic of the time varying parameter is driven by a martingale difference sequence proportional to the score of the conditional likelihood with respect to the parameter of interest.
Although score driven models have been empirically validated multiple times, as for example in [Creal et al., 2013] and [Harvey and Luati, 2014], some questions are still open, partly due to the fact that the class of models has been introduced relatively recently, partly because some issues related to the use of the score have not been fully developed. Specifically, the role of the score as a driving force in the updating equation has not been completely uncovered. In addition, there are currently no theoretically motivated prescriptions on the proportionality coefficient that multiplies the score in the dynamic equation.

2 The family of Score Driven Models

Formally, a stochastic process $\{Y_t\}_{t\in\mathbb{Z}}$, with associated filtration $\mathcal{F}_t := \sigma(Y_l; l \leq t; l \in \mathbb{Z})$, valued in $\mathbb{R}$ is said to be a score driven model of order (p, q) if there exists a process $\{\tilde{\lambda}_t\}_{t\in\mathbb{Z}}$ taking values in $\tilde{\Lambda} \subseteq \mathbb{R}^n$ such that for all $k \in \mathbb{Z}$

$$Y_t|\mathcal{F}_{t-1} \sim \tilde{p}(y_t|\tilde{\lambda}_t, \boldsymbol{\theta})$$
$$\tilde{\lambda}_{t+1} = \gamma + \sum_{i=0}^{q} \alpha_i S_{t-i} s_{t-i} + \sum_{i=0}^{p} \beta_i \tilde{\lambda}_{t-i}$$
$$s_t = \frac{\partial \log \tilde{p}(Y_t|\tilde{\lambda}_t, \theta)}{\partial \tilde{\lambda}_t}$$

for some initial $\bar{\lambda}_1 \in \Lambda \subset \mathbb{R}^n$, where $p, q \in \mathbb{N}$, $\tilde{p}(y_t|\tilde{\lambda}_t, \boldsymbol{\theta})$ is a parametric conditional density, $\tilde{\lambda}_t$ is a filtered value of the true parameter λ_t that governs the stochastic process $\{Y_t\}_{t\in\mathbb{N}}$, $\boldsymbol{\theta} \in \Theta \subseteq R^d$ is the vector of static parameters, and $S_t := S(\tilde{\lambda}_t, \boldsymbol{\theta})$ is a positive measurable scaling function that possibly depends on the filtered time-varying parameter $\tilde{\lambda}_t$ and the static parameter $\boldsymbol{\theta}$.
As the seminal work of [Nelson, 1990] pointed out, understanding the diffusion limit of observation driven models is a way to investigate their theoretical properties. To do so [Nelson, 1990] utilizes a convergence type theorem provided by [Stroock and Varadhan, 1997] while focusing on the famous GARCH(1,1) model, specified by [Bollerslev, 1987] .
The same continuous limit of the GARCH(1,1) is then investigated by [Corradi, 2000] with different modes of convergence of the discrete time interval. Recently, following the ideas of the aforementioned literature, [Buccheri et al., 2021] studied, in the one dimensional setting, conditions under which score driven models of order (1,1) converge in distribution to a stochastic differential equation as the interval between observations goes to zero.

3 Research questions and objectives

Following the works of [Nelson, 1990], [Corradi, 2000] and [Buccheri et al., 2021] there are a number of further theoretical questions that remain to be addressed. In particular we aim to investigate the diffusion limit of scored driven models in an multidimensional setting with different speeds of convergence of the discrete time interval. Also, we aim to study the convergence in distribution to a stochastic differential equation of score driven models of arbitrary order.
In general we aim to uncover theoretical reasons for why the particular structure of the dynamic equation of score driven models allows them to perform

well in applications. An in depth knowledge of the mathematical properties that govern the family of score driven models allows to recognize the applied setting in which they works best and which novel specifications could improve our current state of the art econometric models.

References

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